

ラプラス変換

学籍番号

氏名

$f(t)$	l	a	ε^{-at}	ε^{at}	$\sin \omega t$	$\cos \omega t$	t	$t^2/2$	$f'(t)$	$f''(t)$
$F(s)$	$1/s$	a/s	$1/(s+a)$	$1/(s-a)$	$\omega/(s^2 + \omega^2)$	$s/(s^2 + \omega^2)$	$1/s^2$	$1/s^3$	$sF(s)-f(0)$	$s^2 F(s)-sf'(0)-f''(0)$

問1 $f(t)=\varepsilon^{at}$, $f(t)=t$ のラプラス変換 $F(s)$ を証明せよ。

$$F(s) = \int_0^\infty \varepsilon^{-st} \cdot \varepsilon^{at} dt = \int_0^\infty \varepsilon^{-(s-a)t} dt = \frac{-1}{s-a} \left[\varepsilon^{-(s-a)t} \right]_0^\infty$$

$$= \frac{-1}{s-a} [0 - 1] = \frac{1}{s-a}$$

$$F(s) = \int_0^\infty \varepsilon^{-st} \cdot t dt$$

部分積分 $\int f(t)g'(t)dt = [f(t)g(t)] - \int f'(t)g(t)dt$

より

$$F(s) = \int_0^\infty t \varepsilon^{-st} dt = \left[t \cdot \left(-\frac{1}{s}\varepsilon^{-st}\right) \right]_0^\infty - \int_0^\infty \left(-\frac{1}{s}\varepsilon^{-st}\right) dt$$

$$= 0 + \frac{1}{s^2} \left[\varepsilon^{-st} \right]_0^\infty = -\frac{1}{s^2} [0 - 1] = \frac{1}{s^2}$$

問2 次の関数をラプラス変換せよ。

(1) $f(t)=\varepsilon^{-2t}+3\varepsilon^{5t}$

$$F(s) = \mathcal{L}[\varepsilon^{-2t}] + 3\mathcal{L}[\varepsilon^{5t}] = \frac{1}{s+2} + \frac{3}{s-5}$$

(2) $f(t)=A+B\varepsilon^{-Ct}+D\varepsilon^{-Ft}$

$$F(s) = A\mathcal{L}[1] + B\mathcal{L}[\varepsilon^{-Ct}] + D\mathcal{L}[\varepsilon^{-Ft}] = \frac{A}{s} + \frac{B}{s+C} + \frac{D}{s+F}$$

(3) $f(t)=2\sin 5t - 3\cos \sqrt{2}t$

$$F(s) = 2\mathcal{L}[\sin 5t] - 3\mathcal{L}[\cos \sqrt{2}t]$$

$$= 2 \times \frac{5}{s^2+5^2} - 3 \times \frac{s}{s^2+(\sqrt{2})^2} = \frac{10}{s^2+25} - \frac{3s}{s^2+2}$$

(4) $f(t)=\varepsilon^{-at} \sin \omega t$

$$\mathcal{L}[g(t)\varepsilon^{-at}] = G(s+a)$$

$$F(s) = G(s+a) = \frac{\omega}{(s+a)^2 + \omega^2}$$

(6) $f(t)=t\varepsilon^{-2t} \cos 3t$

$$\mathcal{L}[\cos 3t] = \frac{s}{s^2+3^2} = \frac{s}{s^2+9}$$

$$\mathcal{L}[f(t)] = F(s) \text{ とすると } \mathcal{L}[tf(t)] = -\frac{dF(s)}{ds}$$

$$\left\{ \frac{f(t)}{g(t)} \right\}' = \frac{f'(t)g(t) - f(t)g'(t)}{\{g(t)\}^2}$$

$$\mathcal{L}[t \cos 3t] = -\frac{d}{ds} \left(\frac{s}{s^2+9} \right) = -\frac{s'(s^2+9) - s(s^2+9)'}{(s^2+9)^2}$$

$$= -\frac{s^2+9-2s^2}{(s^2+9)^2} = \frac{s^2-9}{(s^2+9)^2}$$

$$\mathcal{L}[t \cos 3t \cdot \varepsilon^{-2t}] = G(s+2) = \frac{(s+2)^2-9}{((s+2)^2+9)^2}$$

問3 次の関数をラプラス逆変換せよ。

(7) $F(s) = \frac{1}{s-2}$

$$f(t) = \varepsilon^{2t}$$

(8) $F(s) = \frac{s}{s^2+4}$

$$F(s) = \frac{s}{s^2+2^2} \text{ より } f(t) = \cos 2t$$

(9) $F(s) = \frac{2s}{s^2-1}$

$$F(s) = \frac{\alpha}{s+1} + \frac{\beta}{s-1} = \frac{(\alpha+\beta)s + \beta - \alpha}{s^2-1} = \frac{2s}{s^2-1}$$

$$\alpha + \beta = 2, \beta - \alpha = 0 \text{ より } \alpha = \beta = 1$$

$$F(s) = \frac{1}{s+1} + \frac{1}{s-1}$$

$$f(t) = \varepsilon^{-t} + \varepsilon^t$$

(10) $F(s) = \frac{1}{s^2+4s+13}$

$$F(s) = \frac{1}{s^2+4s+13} = \frac{1}{(s+\alpha)^2 + \beta^2} = \frac{1}{s^2+2\alpha s + \alpha^2 + \beta^2}$$

$$2\alpha = 4, \alpha^2 + \beta^2 = 13 \text{ より } \alpha = 2, \beta = 3 (\beta > 0)$$

$$F(s) = \frac{1}{3} \cdot \frac{3}{(s+2)^2 + 3^2}$$

$$f(t) = \frac{1}{3} \varepsilon^{-2t} \sin 3t$$

(11) $F(s) = \frac{s+3}{s(s+1)(s+2)}$

$$F(s) = \frac{s+3}{s(s+1)(s+2)} = \frac{\alpha}{s} + \frac{\beta}{s+1} + \frac{\gamma}{s+2}$$

$$= \frac{(\alpha + \beta + \gamma)s^2 + (3\alpha + 2\beta + \gamma)s + 2\alpha}{s(s+1)(s+2)}$$

$$\alpha + \beta + \gamma = 0, 3\alpha + 2\beta + \gamma = 1, 2\alpha = 3 \text{ より } \alpha = \frac{3}{2}, \beta = -2, \gamma = \frac{1}{2}$$

$$F(s) = \frac{3}{2} \cdot \frac{1}{s} - 2 \cdot \frac{1}{s+1} + \frac{1}{2} \cdot \frac{1}{s+2}$$

$$f(t) = \frac{3}{2} - 2\varepsilon^{-t} + \frac{1}{2}\varepsilon^{-2t}$$

(12) $F(s) = \frac{1}{s^5}$

$$F(s) = \frac{1}{s^2} \Rightarrow f(t) = t$$

$$F(s) = \frac{1}{s^3} \Rightarrow f(t) = \frac{1}{2}t^2$$

$$F(s) = \frac{1}{s^4} \Rightarrow f(t) = \frac{1}{3 \cdot 2 \cdot 1} t^3 = \frac{1}{6}t^3$$

$$F(s) = \frac{1}{s^5} \Rightarrow f(t) = \frac{1}{4 \cdot 3 \cdot 2 \cdot 1} t^4 = \frac{1}{24}t^4$$

問4 以下の微分方程式を解け.

$$(13) y'(t) - 2y(t) = \varepsilon^t \quad (y(0) = 1)$$

$$\mathcal{L}[y'(t) - 2y(t)] = \mathcal{L}[\varepsilon^t]$$

$$sY(s) - y(0) - 2Y(s) = \frac{1}{s-1}$$

$$(s-2)Y(s) = \frac{1}{s-1} + 1$$

$$Y(s) = \frac{1}{(s-1)(s-2)} + \frac{1}{s-2} = \left[\frac{1}{s-2} - \frac{1}{s-1} \right] + \frac{1}{s-2}$$

$$= \frac{2}{s-2} - \frac{1}{s-1}$$

$$y(t) = \mathcal{L}^{-1} \left[\frac{2}{s-2} \right] - \mathcal{L}^{-1} \left[\frac{1}{s-1} \right] = 2\varepsilon^{2t} - \varepsilon^t$$

$$(14) y'(t) - 2y(t) = \varepsilon^{2t} \quad (y(0) = 0)$$

$$\mathcal{L}[y'(t) - 2y(t)] = \mathcal{L}[\varepsilon^{2t}]$$

$$sY(s) - y(0) - 2Y(s) = \frac{1}{s-2}$$

$$Y(s) = \frac{1}{(s-2)^2}$$

$$y(t) = \mathcal{L}^{-1} \left[\frac{1}{(s-2)^2} \right] = \varepsilon^{2t} \mathcal{L}^{-1} \left[\frac{1}{s^2} \right] = \varepsilon^{2t} t$$

$$(15) y'(t) + y(t) = t \quad (y(0) = 0)$$

$$\mathcal{L}[y'(t) + y(t)] = \mathcal{L}[t]$$

$$sY(s) - y(0) + Y(s) = \frac{1}{s^2}$$

$$(s+1)Y(s) = \frac{1}{s^2} \Rightarrow Y(s) = \frac{1}{s^2(s+1)}$$

$$\frac{1}{s^2(s+1)} = \frac{\alpha}{s} + \frac{\beta}{s^2} + \frac{\gamma}{s+1} = \frac{(\alpha+\gamma)s^2 + (\alpha+\beta)s + \beta}{s^2(s+1)}$$

$$\alpha + \gamma = 0, \alpha + \beta = 0, \beta = 1 \quad \text{よ} \quad \alpha = -1, \beta = 1, \gamma = 1$$

$$y(t) = -\mathcal{L}^{-1} \left[\frac{1}{s} \right] + \mathcal{L}^{-1} \left[\frac{1}{s^2} \right] + \mathcal{L}^{-1} \left[\frac{1}{s+1} \right] = -1 + t + \varepsilon^{-t}$$

(16)

$$y''(t) - y'(t) - 2y(t) = 2\varepsilon^t \quad (y(0) = 1, y'(0) = 0)$$

$$\mathcal{L}[y''(t) - y'(t) - 2y(t)] = \mathcal{L}[2\varepsilon^t]$$

$$s^2Y(s) - sy(0) - y'(0) - (sY(s) - y(0)) - 2Y(s) = 2 \cdot \frac{1}{s-1}$$

$$(s^2 - s - 2)Y(s) = s - 1 + \frac{2}{s-1} = \frac{s^2 - 2s + 3}{s-1}$$

$$Y(s) = \frac{s^2 - 2s + 3}{s-1} \cdot \frac{1}{(s-2)(s+1)}$$

$$\frac{s^2 - 2s + 3}{(s-1)(s+1)(s-2)} = \frac{\alpha}{s-1} + \frac{\beta}{s+1} + \frac{\gamma}{s-2}$$

$$s^2 - 2s + 3 = \alpha(s+1)(s-2) + \beta(s-1)(s-2) + \gamma(s-1)(s+1)$$

$$= (\alpha + \beta + \gamma)s^2 - (\alpha + 3\beta)s - 2\alpha + 2\beta - \gamma$$

$$\alpha + \beta + \gamma = 1, \alpha + 3\beta = 2, -2\alpha + 2\beta - \gamma = 3 \quad \text{よ} \quad \alpha = -1, \beta = 1, \gamma = 1$$

$$Y(s) = -\frac{1}{s-1} + \frac{1}{s+1} + \frac{1}{s-2}$$

$$y(t) = -\mathcal{L}^{-1} \left[\frac{1}{s-1} \right] + \mathcal{L}^{-1} \left[\frac{1}{s+1} \right] + \mathcal{L}^{-1} \left[\frac{1}{s-2} \right]$$

$$= -\varepsilon^t + \varepsilon^{-t} + \varepsilon^{2t}$$

$$(17) y''(t) + 4y(t) = 3 \cos t \quad (y(0) = 1, y'(0) = 4)$$

$$\mathcal{L}[y''(t) + 4y(t)] = \mathcal{L}[3 \cos t]$$

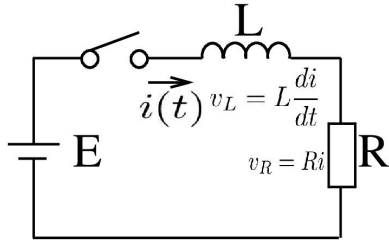
$$s^2Y(s) - sy(0) - y'(0) + 4Y(s) = 3 \cdot \frac{s}{s^2 + 1^2}$$

$$(s^2 + 4)Y(s) = \frac{3s}{s^2 + 1} + s + 4 = \frac{s^3 + 4s^2 + 4s + 4}{s^2 + 1}$$

$$Y(s) = \frac{s(s^2 + 4) + 4(s^2 + 1)}{(s^2 + 1)(s^2 + 4)} = \frac{s}{s^2 + 1} + \frac{4}{s^2 + 4}$$

$$y(t) = \mathcal{L}^{-1} \left[\frac{s}{s^2 + 1} \right] + 2\mathcal{L}^{-1} \left[\frac{2}{s^2 + 2^2} \right] = \cos t + 2 \sin 2t$$

問5 RL直列回路に直流電圧Eを加えたときに流れる電流*i(t)*をラプラス変換法によって求めよ。*t=0*のとき、電流*i(0)=0*とする。



$$E = Ri + L \frac{di}{dt} \Rightarrow E = Ri(t) + Li'(t)$$

$$\mathcal{L}[E] = R\mathcal{L}[i(t)] + L\mathcal{L}[i'(t)]$$

$$\frac{E}{s} = RI(s) + L[sI(s) - i(0)]$$

$$\frac{E}{s} = (R + sL)I(s)$$

$$I(s) = \frac{E}{s} \cdot \frac{1}{R + sL} = E \cdot \frac{\frac{1}{L}}{s(s + \frac{R}{L})}$$

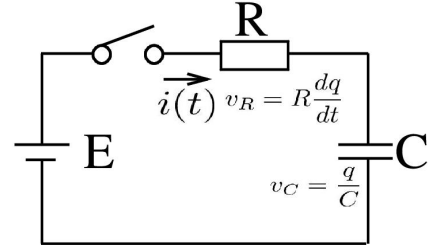
$$\frac{\frac{1}{L}}{s(s + \frac{R}{L})} = \frac{\alpha}{s} + \frac{\beta}{s + \frac{R}{L}} = \frac{(\alpha + \beta)s + \alpha \cdot \frac{R}{L}}{s(s + \frac{R}{L})}$$

$$\alpha + \beta = 0, \alpha \cdot \frac{R}{L} = \frac{1}{L} \text{よ} \text{り} \alpha = \frac{1}{R}, \beta = -\frac{1}{R}$$

$$I(s) = \frac{E}{R} \cdot \frac{1}{s} - \frac{E}{R} \cdot \frac{1}{s + \frac{R}{L}}$$

$$i(t) = \frac{E}{R} \left[\mathcal{L}^{-1} \left[\frac{1}{s} \right] - \mathcal{L}^{-1} \left[\frac{1}{s + \frac{R}{L}} \right] \right] = \frac{E}{R} \left(1 - e^{-\frac{R}{L}t} \right)$$

問6 RC直列回路に直流電圧Eを加えたときに流れる電流*i(t)*をラプラス変換法によって求めよ。*t=0*のとき、電荷*q(0)=0*とする。



$$E = R \frac{dq}{dt} + \frac{q}{C} \Rightarrow E = Rq'(t) + \frac{1}{C}q(t)$$

$$\mathcal{L}[E] = R\mathcal{L}[q'(t)] + \frac{1}{C}\mathcal{L}[q(t)]$$

$$\frac{E}{s} = R[sQ(s) - q(0)] + \frac{1}{C}Q(s)$$

$$\frac{E}{s} = \left(sR + \frac{1}{C} \right) Q(s)$$

$$Q(s) = \frac{E}{s} \cdot \frac{1}{sR + \frac{1}{C}} = E \cdot \frac{\frac{1}{R}}{s(s + \frac{1}{CR})}$$

$$\frac{\frac{1}{R}}{s(s + \frac{1}{CR})} = \frac{\alpha}{s} + \frac{\beta}{s + \frac{1}{CR}} = \frac{(\alpha + \beta)s + \alpha \cdot \frac{1}{CR}}{s(s + \frac{1}{CR})}$$

$$\alpha + \beta = 0, \frac{\alpha}{CR} = \frac{1}{R} \text{よ} \text{り} \alpha = C, \beta = -C$$

$$Q(s) = CE \left(\frac{1}{s} - \frac{1}{s + \frac{1}{CR}} \right)$$

$$q(t) = CE \mathcal{L}^{-1} \left[\frac{1}{s} - \frac{1}{s + \frac{1}{CR}} \right]$$

$$q(t) = CE \left(1 - e^{-\frac{1}{CR}t} \right)$$

$$i(t) = \frac{q(t)}{dt} = CE \left(-1 \cdot \left[-\frac{1}{CR} \right] e^{-\frac{1}{CR}t} \right) = \frac{E}{R} e^{-\frac{1}{CR}t}$$